Kinematics of discretely self-similar spherically symmetric spacetimes

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We summarize the consequences of the twin assumptions of (discrete) self-similarity and spherical symmetry for the global structure of a spacetime. All such spacetimes can be constructed from two building blocks, the "fan" and "splash". Each building block contains one radial null geodesic that is invariant under the self-similarity (self-similarity horizon).

I. INTRODUCTION

Continuous self-similarity (CSS), or homothety, in a spacetime expresses the absence of a preferred length or timescale. As a continuous symmetry, it reduces the number of coordinates on which the metric depends by one. In combination with spherical symmetry, it therefore reduces the Einstein equations to a system of ordinary differential equations, and this is one reason why spacetimes with these combined symmetries have been studied extensively. Besides simplicity, a general reason for studying solutions with symmetries in physics is that attractors of the time evolution flow in phase space usually have more symmetries than generic solutions. For example, in general relativity the end state of gravitational collapse is a Kerr-Newman black hole, which is stationary and axisymmetric.

From a physical point of view, spherically symmetric CSS solutions were studied, among other reasons, because they give rise to naked singularities from regular initial states [1, 2]. It has been suggested that the formation of a naked singularity is always associated with self-similar collapse [3]. These singularities did not seem, however, to be attractors of the evolution flow (but see [4]).

Then Choptuik [5] found a self-similar solution acting as an *intermediate attractor* in gravitational collapse close to the black hole formation threshold. The effects seen by Choptuik with scalar field matter were confirmed by Abrahams and Evans [6] in the collapse of axisymmetric gravitational waves and by Evans and Coleman [7] in spherical fluid collapse with the scale-free equation of state $p = \rho/3$. For a review see [8].

In all these cases the intermediate attractor, or critical solution, is self-similar and contains a naked singularity. The fluid critical solution is CSS, but the scalar field critical solution and the gravitational wave critical solution display a discrete self-similarity (DSS). Such a symmetry had not previously been considered in general relativity (but see [9] for DSS in other sciences).

CSS spherical perfect fluid solutions were generated numerically by Ori and Piran [2]. A family of CSS spherical scalar field solutions was obtained in closed form by Roberts [10]. All such solutions were later classified by Brady [11]. A semi-kinematical classification of CSS spherical null dust spacetimes was carried out by Nolan [12].

Here, we review the purely kinematical consequences (that is, not using the Einstein equations) of self-similarity combined with spherical symmetry. Our methods are generalizations of those in [2, 12]. The results of this paper have already been applied to a classification of all CSS spherical perfect fluid solutions [13], and to the scalar field critical solution, which is DSS [14].

II. GEOMETRIC SELF-SIMILARITY

A. Definition and adapted coordinates

To put our results for spherical spacetimes into a wider context, we first discuss self-similarity without assuming any other symmetry. In particular, we do not yet restrict to spherical symmetry. A spacetime is continuously self-similar (CSS), or homothetic, if there is a vector field ξ^a such that

$$\mathcal{L}_\xi g_{ab} = -2 g_{ab}. \tag{1}$$

The value -2 on the right-hand side is a convention that normalizes ξ^μ, but it must be constant. Any additional structure that covariantly defines an orientable vector field (for example, the 4-velocity vector of a fluid) allows to generalize the concept of continuous self-similarity beyond homothety [15], but we will not consider such generalizations here.

In a CSS solution the integral curves of the homothetic vector field (CSS lines) are geometric objects, and hence provide a geometrically preferred fibration of the spacetime. We choose coordinates such that they are lines of constant coordinates x^i, $i = 1, 2, 3$. The metric can then be written in the form

$$g_{\mu\nu} = e^{-2\tau} \bar{g}_{\mu\nu}, \tag{2}$$

where $x^\mu = \{x^0 \equiv \tau, x^i\}$, and where $\bar{g}_{\mu\nu}$ depends on the x^i but not on τ.

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A spacetime is discretely self-similar (DSS) if there is a conformal isometry Φ of the spacetime such that

$$\Phi_* g_{ab} = e^{-2\Delta} g_{ab}. \tag{3}$$

Δ, the dimensionless scale period, is made unique by taking it to be the smallest positive value for which (3) holds. The value of Δ is then a geometric property of the spacetime. The metric of a DSS spacetime can also be written in the form (2), where $\bar g_{\mu\nu}$ is now allowed to depend periodically on τ with period Δ.

The form (2) of the DSS metric is preserved under coordinates changes of the form

$$\begin{aligned} x'^i &= \varphi^i(\tau, x^j), && (4)\\ \tau' &= \tau + \psi(\tau, x^j), && (5)\end{aligned}$$

where φ^i and ψ are periodic in τ with period Δ. In analogy with CSS lines we call the lines of constant x^i DSS lines, but unlike the CSS lines they are not unique. The non-uniqueness is parameterized by (4).

A closely related type of coordinate system frequently used for CSS or DSS spacetimes is obtained when one replaces τ by $t \equiv -e^{-\tau}$. (The two minus signs are merely a matter of convention. Note that t is not necessarily a time coordinate.) The metric becomes

$$ds^2 = \bar g_{00}\, dt^2 - 2t\, \bar g_{0i}\, dt\, dx^i + t^2\, \bar g_{ij}\, dx^i\, dx^j, \tag{6}$$

where the metric coefficients $\bar g_{\mu\nu}$ are the same as in (2). If one also replaces the x^i by $r^i \equiv (-t)x^i$, the metric becomes

$$\begin{aligned} ds^2 &= (\bar g_{00} + 2\bar g_{0i} x^i + g_{ij} x^i x^j)\, dt^2 \\ &+ 2(\bar g_{0i} + \bar g_{ij} x^j) dt\, dr^i + \bar g_{ij}\, dr^i\, dr^j \\ &\equiv \gamma_{00}\, dt^2 + 2\gamma_{0i}\, dt\, dr^i + \gamma_{ij}\, dr^i\, dr^j. \end{aligned} \tag{7}$$

Note that all coordinates can be spacelike, null, or timelike, as we have not made any assumptions about the sign of the metric coefficients. Note also that in coordinates t and r^i the metric coefficients $\gamma_{\mu\nu}$ still depend only on $x^i = r^i/(-t)$ in the CSS case, and in the DSS case also periodically on $\tau = -\log(-t)$.

It is natural to think of the coordinates x^i and τ as dimensionless, and of t and r^i as having dimension length. In this case one should define $t \equiv -l_0 e^{-\tau}$, where l_0 is a fixed arbitrary length scale, and similarly in other expressions. In the following we do not write l_0 for simplicity.

B. Geometric structure

The decomposition (2) of the metric suggests that we classify elements of the spacetime as "kinematical" or "dynamical" depending on whether they are generated by the conformal factor $e^{-2\tau}$ or the structure of $\bar g_{\mu\nu}$. Some of the effects of the conformal factor can be understood by dimensional analysis. For example, the Kretschmann scalar K, which has dimension (length)$^{-4}$, scales as $e^{4\tau}$ along CSS or DSS lines:

$$K(\tau, x) = e^{4\tau} \bar K(\tau, x), \tag{8}$$

where $\bar K$ depends on x, and in DSS also periodically on τ. ($\bar K$ is not the Kretschmann scalar of $\bar g_{\mu\nu}$.) All other curvature scalars scale similarly, according to their dimension. Therefore $\tau = \infty$, for any x^i, is *generically* a scalar curvature singularity of any self-similar spacetime. We call it the "kinematical singularity". However, $\bar K = 0$ or ∞ can occur on an isolated CSS or DSS line, say $x = 0$, and in this case the simultaneous limit $x \to 0$, $\tau \to \pm\infty$ may depend on the curve $x(\tau)$ on which the limit is reached. Furthermore, dimensional analysis may be misleading for frame components of the curvature. An example of this in spherical symmetry is described below (a fan with $q < -1$), where a frame component blows up at $\tau = -\infty$.

Because the natural dimension for an affine parameter is length, dimensional analysis suggests that the affine parameter scales as $e^{-\tau}$, so that DSS lines are finite as $\tau \to \infty$, but infinitely extended as $\tau \to -\infty$. This is consistent with the fact that $\tau = -\infty$ generically corresponds to infinite area radius r. We therefore call $\tau = -\infty$ the "kinematical infinity". However, Lake and Zannias [16] show that it is possible to have complete null CSS lines. We have found an example of this in spherical symmetry (a fan with $q = -1$), where the affine parameter is τ, rather than $e^{-\tau}$.

As we assume that the spacetime is *globally* fibrated by CSS or DSS lines, its global manifold structure must be the product of a 3-manifold (with coordinates x^i) and the τ line, the cylinder $M = \Sigma_3 \times R$. Note that the geometry need not correspond to the manifold structure. The kinematical singularity, for example, can be a single point, a line, a 2-dimensional surface or a 3-dimensional surface.

In addition to the kinematical singularity and infinity, singularities and infinities can also be generated by the structure of $\bar g_{\mu\nu}$. We shall call these "dynamical". If we define a singularity as an obstruction to continuing the spacetime, and consider only inextendible spacetimes, by definition the boundaries of the spacetime are either singularities or infinities. The kinematical boundaries $\tau = \pm\infty$ are copies of Σ_3. The dynamical boundaries of the spacetime must be invariant under the self-similarity, and so the dynamical boundary has the manifold structure $\partial\Sigma_3 \times R$.

A hypersurface that is invariant under the self-similarity can in general be deformed continuously into another such hypersurface. On the other hand, this is generically not true for an invariant *null* hypersurface. Such a hypersurface is therefore a geometrical object and is called a self-similarity horizon (SSH).

III. SPHERICAL SYMMETRY

A. Spherical symmetry without self-similarity

For the remainder of the paper we restrict our discussion to spherical symmetry. The four-dimensional spacetime is the product of a two-dimensional spacetime with coordinates τ and x (the reduced spacetime) and a round two-sphere of area $4\pi r^2$. We write the most general metric adapted to spherical symmetry in the form

$$ds^2 = e^{-2\tau}\left(A\,d\tau^2 + 2B\,d\tau\,dx + C\,dx^2 + F^2\,d\Omega^2\right), \quad (9)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ is the metric on the unit 2-sphere, and where A, B, C and F are functions of τ and x. The explicit factor $e^{-2\tau}$ has been written in anticipation of self-similarity. In the following we assume that the signature is $(-,+,+,+)$, and that the metric is non-degenerate except possibly at the boundaries, so that $AC - B^2 < 0$. Note that we do not mean to imply that τ is timelike and x spacelike: we have not fixed the signs of A, B and C. We choose $F \geq 0$ by convention. For simplicity, we assume that A, B, C, F are all C^2, so that the Riemann tensor is C^0. (Note that a complete set of Einstein equations can be constructed from only first derivatives of A, B, C and second derivatives of F.)

The area radius $r \equiv e^{-\tau}F$ is a scalar in the reduced manifold. A second geometrical scalar is the Hawking mass m defined by $1 - 2m/r \equiv (\nabla r)^2$. From m and r we can define the dimensionless scalar $\mu \equiv 2m/r$. A spherical surface (point in the reduced spacetime) where $\mu \geq 1$ is a closed trapped surface, and one where $\mu = 1$ is an apparent horizon.

B. Spherical symmetry and self-similarity

We now restrict to self-similarity, discussing the continuous and discrete symmetries separately.

The metric (9) is CSS with homothetic vector $\partial/\partial\tau$ if and only if A, B, C and F depend only on x. (The same then holds for μ.) The CSS lines are the lines of constant x on the reduced manifold. All coordinate systems with this property are related by coordinate transformations of the form

$$x' = \varphi(x), \qquad \tau' = \tau + \psi(x). \quad (10)$$

The first of these just relabels the CSS lines. This is similar to a change of radial coordinate in a coordinate system for Schwarzschild (for example from the area radius to the isotropic radial coordinate). The second changes the τ slicing, and can be used to turn τ into a global coordinate. This is similar to a change of slicing in Schwarzschild, for example from the Schwarzschild to the Painlevé-Güllstrand time.

CSS lines are timelike for $A < 0$, null for $A = 0$, and spacelike for $A > 0$. Therefore the sign of A is a geometric property of a spacetime region. The lines of constant τ are timelike for $C < 0$, null for $C = 0$, and spacelike for $C > 0$. This is merely a property of the coordinate system, and can be changed by a transformation of the form (10).

The metric (9) is DSS with scale period Δ, with the discrete conformal isometry $\Phi : (\tau, x) \to (\tau+\Delta, x)$, if and only if $A(\tau, x) = A(\tau + \Delta, x)$, and similarly for B, C and F. (The same then holds for μ.) All coordinate systems adapted to DSS in this way are related by coordinate transformations of the form

$$x' = \varphi(\tau, x), \qquad \tau' = \tau + \psi(\tau, x). \quad (11)$$

where φ and ψ are periodic in τ with period Δ.

DSS lines can be moved around by a coordinate transformation (as well as relabeled). It is clear that we cannot transform a DSS line with $A > 0$ for all τ into one with $A < 0$, or vice versa. If the sign of A changes with τ along a DSS line, the line can be moved ("straightened out") until A is either strictly positive, strictly negative or zero for all τ. In the following we always assume that the spacetime is covered by one global coordinate patch with this property. This can always be done, although for numerical calculations it may be better to use local patches in which the metric can be simplified.

In such a coordinate system, the sign of A is a geometric property of a region of spacetime just as in CSS. This restriction does not make the DSS lines with $A > 0$ or $A < 0$ rigid, because these are just inequalities, but it does make the $A = 0$ lines rigid: we cannot deform such a line without making parts of it spacelike or timelike. $A = 0$ lines are radial null geodesics that are invariant under the self-similarity. In the full spacetime they correspond to spherical self-similarity horizons. In the following, we shall focus on isolated SSHs. When the SSH is not isolated, A need not vanish there. An example is given in Sec. V.

DSS lines and lines of constant τ are normal to each other if and only if $B = 0$. The geometric interpretation of the sign of B is more complicated, and will be deferred to Sec. IV A.

C. Boundaries of the reduced spacetime

The 4-dimensional spacetime is fibrated by CSS lines or DSS lines, and so the *reduced* spacetime is *foliated* by CSS or DSS lines, all extending from $\tau = -\infty$ to $\tau = \infty$. The manifold structure of the reduced spacetime is therefore a rectangle, the τ line times an interval in x. In the DSS case, we make the assumption that both dynamical boundaries, as well as any SSHs, are DSS lines $x = \text{const}$. These are only local restrictions on the coordinate system. We then have two kinematical boundaries $\tau = \pm\infty$ and two dynamic boundaries $x = x_{\min,\max}$. There are three types of dynamical boundary:

A dynamical infinity is a dynamical boundary where radial geodesics reach infinite affine parameter with fi-

nite τ. Most naturally this coincides with $F=\infty$, but finite F is also conceivable, corresponding to a wormhole with topology $R^2\times S^2$ that stretches to infinity in the x-direction with finite circumferences $2\pi r$.

A dynamical singularity is a dynamical boundary where radial null geodesics end in infinite curvature at finite affine parameter, at finite τ. A dynamical singularity can be central, with $F=0$, or conceivably be non-central, with $F\neq 0$. As mentioned in the discussion of the general (non-spherical case), any weak singularities through which the spacetime can be continued would be considered interior points.

A regular center is given by $F=0$ with the additional condition $\mu\sim F^2$ as $F\to 0$. This is equivalent to $m\sim r^3$, or the absence of a defect angle. This boundary of the reduced spacetime is of course not a boundary of the full spacetime but just the central world line.

$F(x,\tau)=0$ or ∞ always corresponds to one of these three boundaries, and so cannot occur except at the dynamical boundaries. Therefore, as $r=e^{-\tau}F(\tau,x)$, the kinematical boundary $\tau=\infty$ is always central and the kinematical boundary $\tau=-\infty$ always has infinite area. However, infinite area radius r does not necessarily mean an infinity in the sense of infinite affine parameter, as we shall see below in Sec. IV F 3.

IV. SELF-SIMILARITY HORIZONS

A. Radial null geodesics

In this section we investigate the spacetime structure near isolated self-similarity horizons in spherical symmetry. Without loss of generality, we assume in the following that the isolated SSH under consideration is at $x=0$.

The radial null geodesics of the metric (9) are given by

$$\frac{dx}{d\tau}=C^{-1}\left(-B\pm\sqrt{B^2-AC}\right)\equiv f_\pm(x,\tau). \quad (12)$$

If we cover the entire spacetime with a global coordinate patch, then the two signs in the equation above correspond to the two global families of null geodesics that cross-hatch the reduced spacetime. We shall refer to these as $+$ and $-$ lines. In the conformal diagram they are the left and right moving null lines (where of course left and right are purely conventional names).

The radial null geodesics can also be written as

$$\frac{dx}{d\tau}=C^{-1}\left(-B\pm\frac{B}{|B|}\sqrt{B^2-AC}\right)\equiv f_{u,v}(x,\tau). \quad (13)$$

We refer to first of these as lines of constant u, and the second as lines of constant v. In the limit $A\to 0$ these two families become

$$\frac{dx}{d\tau} = f_u=-\frac{A}{2B}\left[1+O(AC/B^2)\right], \quad (14)$$

$$\frac{d\tau}{dx} = \frac{1}{f_v}=-\frac{C}{2B}\left[1+O(AC/B^2)\right]. \quad (15)$$

Note that as $A\to 0$, we must have $B\neq 0$, because in any regular global coordinate system we have $B^2-AC>0$. The SSH $x=0$ is clearly a u line (solution of the upper equation), and so the u lines are parallel to this particular horizon. Without loss of generality, we set $u=0$ on the SSH. The v lines cross the horizon.

The classification of radial null geodesics as $+$ and $-$ lines is global, and the classification as u and v lines is local to a SSH. Their relation depends on the sign of B: for $B>0$, the u lines are the $+$ lines, while for $B<0$, they are the $-$ lines. Therefore if B does not change sign between two isolated SSHs, they are parallel in the sense that both are $+$ or both are $-$. In the other case, a neigbouring $+$ and $-$ SSH cannot intersect at finite values of their affine parameters, but they can meet at their endpoints, thus forming a "corner" in the reduced spacetime. This idea will be taken up again in Sec. V.

B. SSHs as attractors

The SSH $x=0$ is a stationary point of the dynamical system $dx/d\tau=f_u$. For $xf_u(x)<0$, $|x|\to 0$ as $\tau\to\infty$, while for $xf_u(x)>0$, $|x|\to 0$ as $\tau\to-\infty$. If the metric is C^1 at the SSH, then $A=O(x)$, and therefore $f_u=O(x)$. To see that the attracting fixed point $x=0$ is actually reached, note that (assuming C^2 for simplicity)

$$f_u(\tau,x)\equiv f_1(\tau)x+O(x^2), \qquad f_1(\tau)\equiv\bar f_1+\tilde f_1(\tau), \quad (16)$$

where $\bar f_1$ is the average value of the periodic function $f_1(\tau)$. Then

$$[1+O(x)]\ln x=\bar f_1\tau+\int\tilde f_1(\tau)\,d\tau, \quad (17)$$

where the integral is periodic. A similar calculation holds if the leading order in f is higher than $O(x)$.

We call the SSH a "splash" if it attracts u lines as $\tau\to\infty$, and a "fan" if it attracts them as $\tau\to-\infty$. The motivation for these names will become clear below when we investigate the spacetime structure near the SSH. If $f_u(x)$ has the same sign on both sides of $x=0$, the SSH attracts u lines as $\tau\to\infty$ from one side, and as $\tau\to-\infty$ from the other. We then call it half a fan from one side and half a splash from the other.

In a splash, the coordinate location $(x=0,\tau=\infty)$ is intersected by a range of u lines, and so it cannot be a point, but must be a line in the reduced spacetime. The same is true for the coordinate location $(x=0,\tau=-\infty)$ in a fan.

C. Double null coordinates

We now show that the line $x=0,\tau=\pm\infty$ is null by explicitly constructing double null coordinates on the reduced spacetime. We assume that the metric is regular

and (for simplicity) C^2 at the SSH and that the SSH is isolated. The metric near the SSH then has the form

$$ds^2 = e^{-2\tau}\Big[\big(a(\tau)x + O(x^2)\big)\,d\tau^2 + 2\,(b(\tau)+O(x))\,d\tau\,dx + (c(\tau)+O(x))\,dx^2 + \big(f(\tau)^2+O(x)\big)\,d\Omega^2\Big]. \quad (18)$$

In Appendix A we show that by a coordinate transformation of the type (11) we can bring the metric into the unique form where $a(\tau) = 4q$, $b(\tau) = 1$ and $c(\tau) = 0$. Then

$$ds^2 = e^{-2\tau}\Big[\big(4qx + O(x^2)\big)\,d\tau^2 + (2+O(x))\,d\tau\,dx + O(x)\,dx^2 + \big(f(\tau)^2+O(x)\big)\,d\Omega^2\Big]. \quad (19)$$

Note that $f_u \simeq -2qx$ as $x \to 0$ and so the SSH is a splash for $q > 0$ and a fan for $q < 0$.

We now define a new spacetime, the "q-metric", by dropping all higher order terms from the approximation:

$$ds^2 = e^{-2\tau}\left(4qx\,d\tau^2 + 2\,d\tau\,dx + f(\tau)^2\,d\Omega^2\right). \quad (20)$$

Its region $|x| < \epsilon$ for some small constant ϵ is an approximation to a generic spherically symmetric DSS spacetime near an isolated SSH, but the q-metric is simple enough to be studied exactly.

For $q \neq 0, -1$, we define a pair of null coordinates u and v by

$$u = \frac{1}{p}xe^{2q\tau}, \quad (21)$$

$$v = e^{-2p\tau}, \quad (22)$$

where $p \equiv q+1$. The metric in coordinates u and v is

$$ds^2 = -du\,dv + f^2(\tau)\,v^{1/p}\,d\Omega^2. \quad (23)$$

The definitions of u and v can be inverted to give

$$x = puv^{q/p}, \quad (24)$$

$$\tau = -\frac{\ln v}{2p}. \quad (25)$$

The range of coordinates in the q-metric is $-\infty < u < \infty$ and $0 < v < \infty$. As an approximation to generic SSHs, this metric is good only for $|x| < \epsilon$, which translates to

$$|u| < \frac{\epsilon}{p}v^{-q/p}. \quad (26)$$

In order to draw the conformal diagram, we compactify the metric (23) by setting $v = (\tan V)^{-p/q}$ and $u = (x_0/p)\tan U$ for a constant x_0. This gives

$$ds^2 = \omega^2\left(\frac{x_0}{q}dU\,dV + f(\tau)^2\sin^2 V\cos^2 U\,d\Omega^2\right) \quad (27)$$

$$\omega^2 = (\sin V)^{-2-1/q}(\cos V)^{1/q}(\cos U)^{-2} \quad (28)$$

for $0 \le V \le \pi/2$ and $-\pi/2 \le U \le \pi/2$. With

$$x = x_0\frac{\tan U}{\tan V}, \qquad \tau = \frac{1}{2q}\ln\tan V \quad (29)$$

$x = 0$ has two branches: $U = 0$ and $V = \pi/2$. Therefore the SSH $x = 0$ is bifurcate in the form of a T. $U = \pm V$ corresponds to $x = \pm x_0$, for example the limits $x = \pm\epsilon$ of the validity of our approximation.

D. Geodesics near the SSH

To understand the geometry of the $V = \pi/2$ branch of the SSH, we now investigate the behaviour of the area radius r, the affine parameter and the spacetime curvature along geodesics running into $V = \pi/2$.

Like the spacetimes it approximates, the q-metric is exactly self-similar. In the CSS case the homothetic vector is

$$\frac{\partial}{\partial\tau} = 2qu\frac{\partial}{\partial u} - 2pv\frac{\partial}{\partial v}. \quad (30)$$

However, the q-metric also has the exact Killing vector $\partial/\partial u$, which a generic SSH does not have. The presence of this Killing vector means also that the SSH of the q-metric is not isolated, as every line $u = \text{const}$ is a SSH.

Without loss of generality we assume that the geodesics are in the equatorial plane. Spherical symmetry and the null Killing vector give rise to the conserved quantities

$$L \equiv \left(\frac{\partial}{\partial\varphi}\right)^a u_a = f^2 v^{1/p}\,\dot\varphi, \qquad E \equiv -2\left(\frac{\partial}{\partial u}\right)^a u_a = \dot v, \quad (31)$$

where u^a is the tangent vector to the geodesic, and a dot denotes the derivative with respect to the affine parameter. The norm

$$\kappa \equiv u^a u_a = -\dot u\dot v + f^2(\tau)v^{1/p}\dot\varphi^2 = 0, 1, -1 \quad (32)$$

is also conserved. As $\dot v = E$ is constant, v is an affine parameter for all 4-dimensional geodesics, except for the $v = \text{const}$ radial null geodesics. Defining the impact parameter $D \equiv L/E$, we can integrate the geodesics to give

$$\varphi - \varphi_0 = D\int_0^v f^{-2}v^{-1/p}\,dv \simeq \frac{pD}{qf^2}\,v^{q/p}, \quad (33)$$

$$u - u_0 = D^2\int_0^v f^{-2}v^{-1/p}\,dv - \frac{\kappa}{E^2}\,v \simeq \frac{pD^2}{qf^2}\,v^{q/p} - \frac{\kappa}{E^2}\,v, \quad (34)$$

where the second equality in each case holds exactly in the CSS case, where $f(\tau)$ is constant, and approximately in the DSS case.

In the CSS case, the homothety gives rise to the quantity

$$J \equiv \left(\frac{\partial}{\partial\tau}\right)^a u_a = -qu\dot v + pv\dot u. \quad (35)$$

It is straightforward to check that for all geodesics this correctly obeys

$$\dot{J} = -\kappa. \tag{36}$$

Therefore it was consistent to use the Killing symmetry, rather than the homothetic symmetry, in order to integrate the geodesic equations.

E. Curvature

From the results of Appendix B, the components of the 4-dimensional Ricci tensor in a parallely propagated null tetrad and the Ricci scalar of the q-metric (for $q \neq 0, -1$) are

$$R_{22} = \frac{2q+1}{2p^2} v^{-2}, \tag{37}$$

$$R_{33} = R_{44} = r^{-2}, \tag{38}$$

$$R = 2r^{-2}. \tag{39}$$

Therefore, $r = 0$ is a curvature singularity for all values of q, and $v = 0$ is a curvature singularity for $q \neq -1/2$. Because

$$v = (\tan V)^{-\frac{p}{q}}, \qquad r = f(\tau)\,(\tan V)^{-\frac{1}{2q}}, \tag{40}$$

for $q > 0$ there is a curvature singularity at $V = \pi/2$, for $-1 < q < 0$ there is a curvature singularity at $V = 0$, and for $q < -1$, there are curvature singularities at both $V = 0$ and $V = \pi/2$. In this last case, the curvature singularity at $V = \pi/2$ is non-scalar at $r = \infty$ and is not predicted by dimensional analysis: R vanishes there, but R_{22} blows up.

The 4-dimensional Riemann tensor of any spherically symmetric spacetime is constructed from the 2-dimensional Riemann tensor of the reduced spacetime, plus derivatives of r. Any curvature singularity in the q-metric that arises from derivatives of r has a counterpart in the SSH it approximates. However, we must keep in mind that the 2-dimensional curvature of the reduced spacetime is generically nonzero, and blows up at $\tau = \infty$, while the q-metric approximates it as zero. But we have already identified $\tau = \infty$ as a curvature singularity because $R \sim r^{-2}$.

F. Nature of the boundary $V = \pi/2$

We now have all the tools we need to characterise the neighbourhood $|x| < \epsilon$ of the SSH when $q, p, f(\tau) \neq 0$. The special cases $q = -1$, $q = 0$ and $f(\tau) = 0$ need to be considered separately.

1. *Generic case $q > 0$: splash*

$V = \pi/2$ corresponds to $(x = 0, \tau = \infty)$, $r = 0$ and $v = 0$. It is a null central scalar curvature singularity, reached by all geodesics except the radial null geodesics v = const at finite affine parameter. The values of u, θ and φ also converge as the singularity is reached, and so the singularity is a 3-dimensional null surface, rather than a line, in the 4-dimensional spacetime. A splash is illustrated in Fig. 1.

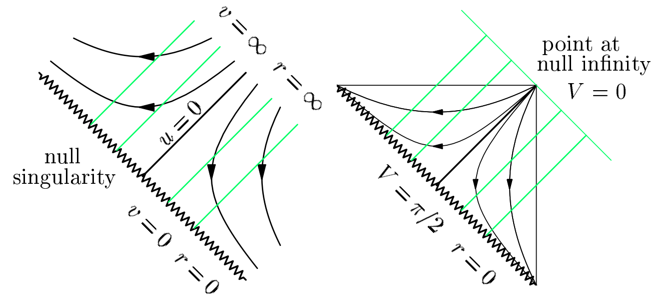

FIG. 1: A splash. The conformal diagram on the right has been obtained from the one on the left by compactification. The lines with arrows are DSS lines with the arrow pointing towards increasing τ. The thin parallel lines are radial null geodesics u = const. The central one $u = 0$ is also the DSS line $x = 0$.

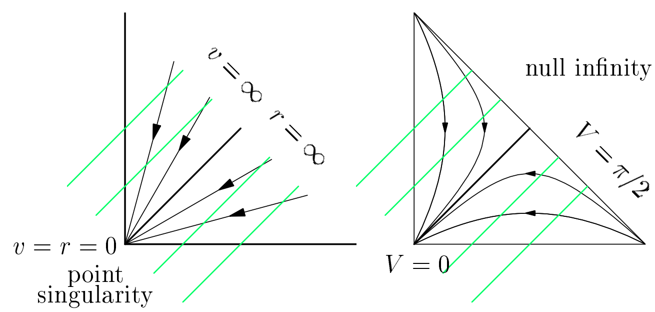

FIG. 2: A fan. The conformal diagram on the right has been obtained from the one on the left by compactification. The lines with arrows are DSS lines with the arrow pointing towards increasing τ. The thin parallel lines are radial null geodesics u = const. The central one $u = 0$ is also the DSS line $x = 0$.

2. *Generic case $-1 < q < 0$: regular fan*

$V = \pi/2$ corresponds to $(x = 0, \tau = -\infty)$, $r = \infty$ and $v = \infty$. It is reached by all null geodesics at infinite affine parameter v and finite values of u, θ and φ. It is missed by all spacelike and timelike geodesics. All curvature scalars and frame components vanish there. In this sense it has the same structure as the null infinity of Minkowski spacetime. A regular fan is illustrated in Fig. 2.

3. Generic case $q < -1$: singular fan

$V = \pi/2$ corresponds to $(x = 0, \tau = -\infty)$, $r = \infty$ and $v = 0$. The curvature scalars vanish but the frame component R_{22} blows up there, and all geodesics end at finite affine parameter. This part of the "kinematical infinity" is therefore in fact a (non-scalar) curvature singularity. From (37) we see that R_{22} is negative for $q < -1/2$. As R_{11} vanishes and the other frame components can be neglected as $x \to 0$, the energy density is therefore negative for any timelike observer. Therefore, although it is kinematically allowed, this spacetime cannot arise with reasonable matter content.

4. Special case $q = -1$: regular fan

This implies $p = 0$ and so must be treated specially. Double null coordinates are now

$$u = xe^{-2\tau}, \tag{41}$$

$$v = -2\tau \tag{42}$$

and the q-metric is

$$ds^2 = -du\,dv + e^v f^2 d\Omega^2. \tag{43}$$

The inverse coordinate transformation gives

$$x = ue^{-v}, \qquad \tau = -\frac{v}{2}, \tag{44}$$

and so $x \to 0$ along radial null geodesics $u =$ const as $\tau \to -\infty$, which is also $r \to \infty$ and $v \to \infty$. The analysis of non-radial, non-null geodesics gives (assuming CSS for simplicity) gives

$$\varphi - \varphi_0 = -\frac{D}{f^2}\,e^{-v}, \qquad u - u_0 = -\frac{D^2}{f^2}\,e^{-v} - \frac{\kappa}{E^2}\,v. \tag{45}$$

Therefore $v = \infty$ has the same structure as Minkowski null infinity, and so the SSH is a regular fan. The only qualitative difference to the $-1 < q < 0$ fan is that the outgoing radial null geodesic $u = 0$ is infinitely extended at both ends.

5. Special case $q = 0$: regular fan, splash or half of each

We have assumed that $A(\tau, x)$ is generic in the sense that near $x = 0$, $A = a(\tau)x + O(x^2)$ with $a \neq 0$. The special case $a(\tau) = 0$ and hence $q = 0$ corresponds to the $O(x)$ term being absent. As we are investigating isolated SSHs, $A = 0$ is not a sufficiently good approximation, and we must include the first non-vanishing order in A. As an example, we consider

$$f_u(x, \tau) \equiv f_m(\tau)x^m + O(x^{m+1}), \qquad f_m(\tau) \equiv \bar{f}_m + \tilde{f}_m(\tau), \tag{46}$$

where $m = 1$ is the generic case that we considered before, and we now consider $m > 1$. Then

$$-\left[1 + O(x)\right]\frac{x^{-(m-1)}}{m-1} \simeq \bar{f}_m\tau + \int \tilde{f}_m(\tau)\,d\tau. \tag{47}$$

Consider the case where $m > 1$ is an integer. If it is odd, $x \to 0$ as $f_m\tau \to \infty$, and so we have a fan for $f_m < 0$ and a splash for $f_m > 0$. If m is even, then $x \to 0$ as $\mathrm{sign}(x) f_m \tau \to \infty$, and so we have half a fan on one side and half a splash on the other.

To see if the affine parameter on u lines is finite or infinite as $x \to 0$, we assume CSS for simplicity. From the definition (35) of J, the form (9) of the metric, and the approximation (14) to the u lines, we find

$$J \simeq -e^{-2\tau} B(x)\dot{x}. \tag{48}$$

On null geodesics J is conserved. As the metric is regular at $u = 0$, B cannot vanish, and so is approximately constant as $x \to 0$. We then have

$$\frac{d\lambda}{d\tau} \sim e^{-2\tau}\frac{dx}{d\tau} \sim e^{-2\tau}|\tau|^{-\frac{m}{m-1}} \tag{49}$$

For all $m > 1$, $\lambda(\tau)$ converges as $\tau \to \infty$ and diverges as $\tau \to -\infty$. Therefore the affine parameter λ converges at $x = 0$ if and only if $r \sim e^{-\tau}$ converges. This means that the SSH is either a splash or a regular fan, but never a singular fan. Note that for a dynamical boundary which is a null infinity, the metric cannot have the form (49). An example is found in App. C.

6. Special case $f(\tau) = 0$

Until now we have assumed that $F(x, \tau)$ does not vanish or blow up at the horizon. Even if it does, our analysis of double null coordinates and radial geodesics in the q-metric is unchanged. For simplicity we assume CSS again, so that $F = F(x)$, and as an example we consider $F(x) \simeq f_n x^n$, where n is not necessarily an integer. The q-metric becomes

$$ds^2 = -du\,dv + f_n^2\left(puv^{q/p}\right)^{2n} v^{1/p}\,d\Omega^2. \tag{50}$$

The branch $u = 0$ of the SSH now has $r = 0$ for $n > 0$ and $r = \infty$ for $n < 0$. The curvature of this metric is given in appendix B. The result is that the branch $u = 0$ of the SSH is a curvature singularity for all $n \neq 0$. In our terminology, the SSH is a dynamical singularity and forms one end of the range of x.

V. FANS AND SPLASHES AS BUILDING BLOCKS

We now show how the conformal diagram of the reduced spacetime can be constructed from fans and

splashes. We shall refer to + and − fans and splashes, according to if the u lines are the + or the − lines.

Consider two neighbouring isolated SSHs x_1 and x_2, so that A vanishes for $x = x_1$ and $x = x_2$ and is either strictly positive or strictly negative in between. If B does not change sign between the two SSHs, then they are one fan and one splash, either both + or both −. An example of this appears near the top of Fig. 3. There x_f is a + fan and x_h, to the future of x_f, is a + splash. If B does change sign between two neighbouring SSHs, then they are either two fans or two splashes. One is + and one is −. An example of this appears toward the right of Fig. 3. x_p is a − fan, to the past of the + fan x_f.

As every SSH is an attractor of one type of radial null geodesic in one direction, and as there are no other fixed points of $dx/d\tau = f$, almost all points on the kinematical boundaries are part of a fan or splash, and so all have the same value of x. In Fig. 3 these values are x_f, x_p and x_h. By definition, the two dynamical boundaries are also at constant values of x. In Fig. 3 these are x_c and x_s, a regular center and a singularity. All intermediate values of x are bunched up in isolated points on the boundary of the conformal diagram.

Because the kinematic boundaries of the conformal diagram are provided by the fans and splashes, all kinematically possible Penrose diagrams of self-similar spacetime diagrams can be enumerated by stringing together, like dominoes, fans and splashes (or sometimes half-fans and half-splashes). As basic building blocks one could pick the half-fan and the half-splash, and one could formalize a set of rules that govern how these can be combined.

While the example in Fig. 3 gives clear examples of fans and splashes, they are harder to spot in spacetimes with simpler conformal diagrams. We give two examples of this. Fig. 4 contains only a single building block, a half-fan. Its SSH is at the same time a dynamical infinity. Fig. 5 illustrates a spacetime with no fans or splashes. The kinematical infinity and singularity are each reduced to a point, and the dynamical boundaries are both singularities.

If a spherical CSS spacetime has additional symmetries, our classification in terms of fans and splashes is still applicable, but no longer unique (or very useful). We give two examples. Fig. 6 illustrates the flat Friedmann solution. It is CSS for any perfect fluid matter with the linear equation of state $p = c_s^2\rho$. This is shown by writing the metric in the form

$$ds^2 = e^{-2\tau}\left[(-1+\eta^2x^2)d\tau^2 - 2\eta x\,dx\,d\tau + dx^2 + x^2\,d\Omega^2\right], \tag{51}$$

where $\eta = (1+3c_s^2)/3(1+c_s^2)$. Clearly there is a fan at $x = 1/\eta$. The big bang is a dynamical singularity in our classification, with the exception of the point at the origin of spherical symmetry, which is the kinematical singularity. But because the solution has additional $E(3)$ translation invariance, every fluid world line can be considered as the center of spherical symmetry. The distinction between the kinematical and dynamical singularity is then completely artificial. For the same reason there is a spherical SSH through every point of the spacetime.

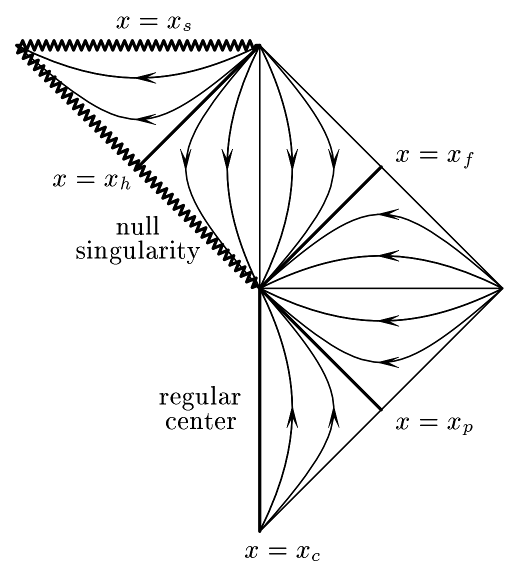

FIG. 3: Penrose diagram for a self-similar spacetime with a regular center at $x = x_c$, two fans at $x = x_p$ and $x = x_f$, and one splash at $x = x_h$. Because the spacetime contains at least one splash, the kinematical singularity is extended and null. Because it contains just one splash, the dynamical singularity $x = x_s$ is spacelike. This spacetime occurs in CSS with scalar field matter, compare Fig. 22 of [2].

Fig. 7 illustrates the scalar field solution discussed by Hayward [17], with the metric

$$ds^2 = e^{-2\tau}[2\,d\tau^2 - 2\,dx^2 + d\Omega^2] \tag{52}$$

and scalar field $\sqrt{4\pi G}\phi = x$. The spacelike homothetic vector $\partial/\partial\tau$ is evident. However, $\partial/\partial x$ is a timelike Killing vector, and any linear combination of $\partial/\partial\tau$ and $\partial/\partial x$ with constant coefficients is also a homothetic vector. In particular, the two linear combinations $\partial/\partial x \pm \partial/\partial\tau$ are null homothetic vectors. Therefore, there are no isolated SSHs, and two SSHs cross in every point of the reduced spacetime. In Fig. 7 we have marked the CSS lines of $\partial/\partial\tau$. In that view, the spacetime diamond is made up from two disjoint half-splashes.

Recall that $A = 0$ at any isolated SSH in CSS. The Hayward spacetime shows that this need not be true if the SSH is not isolated. The reason is that now CSS lines with arbitrary directions go through every point. None of these directions is preferred, but of course only one is null.

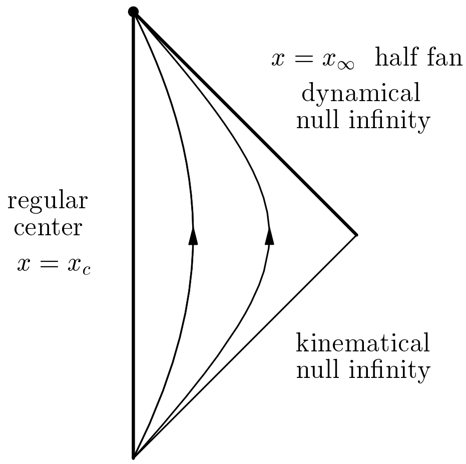

FIG. 4: Penrose diagram for a self-similar spacetime with a regular center at $x = x_c$, and half a fan which is also a dynamical (null) singularity at $x = x_\infty$. This spacetime occurs in CSS with scalar field matter, Class I of [11], with coordinates $x_c = 0$ and $x_\infty = \infty$. The metric is given in App. C.

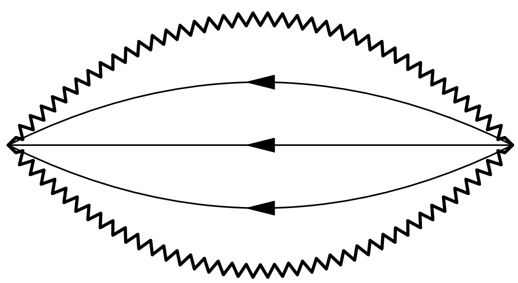

FIG. 5: Penrose diagram for a spherically symmetric solution without any SSH. Here the kinematical infinity and kinematical singularity each consist of one point in the reduced spacetime. The rest of the boundary consists of two spacelike (central) dynamical singularities (both linked to the kinematical singularity). This spacetime can be realized dynamically with perfect fluid matter [13].

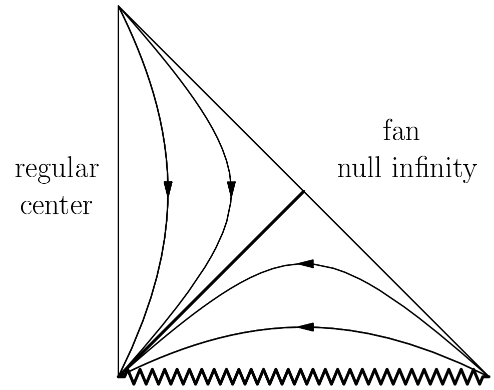

FIG. 6: Penrose diagram for the flat Friedmann solution. Note that because of translation invariance, any point in space can be chosen as the regular center.

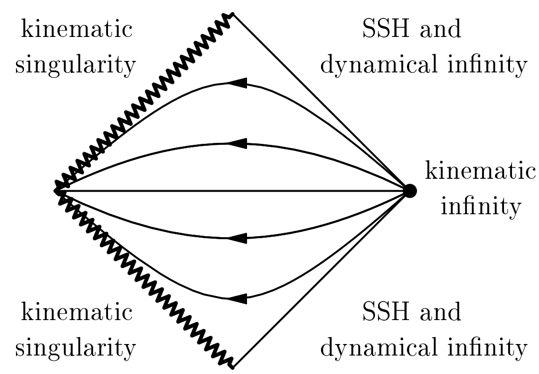

FIG. 7: Penrose diagram for the Hayward solution. The integral curves of the homothetic vector $\partial/\partial\tau$ are shown, but these CSS lines are not unique because the spacetime has two linearly independent homothetic vector fields.

VI. CONCLUSIONS

After a brief discussion of geometric self-similarity in general, we restricted attention to spacetimes that are both spherically symmetric and self-similar. Our investigation has been kinematical in that it uses only those symmetries but not the Einstein equations.

As is well-known, every spherical spacetime can be written as the product of a reduced 1+1-dimensional spacetime with a spacelike 2-sphere at each point. Continuous self-similarity provides a preferred foliation of the reduced spacetime by the integral curves of the homothetic vector field, or CSS lines. In DSS, a foliation by DSS lines is not unique, but it can be restricted enough to serve essentially the same purpose. This, combined with dimensional analysis, suggested a heuristic classification of the boundaries of the reduced spacetime into a kinematical infinity $\tau = -\infty$ where all DSS lines begin, a kinematical singularity $\tau = \infty$ where they end, and two dynamical boundaries $x = x_{\min}$ and $x = x_{\max}$ which *are* DSS lines. Each of the two dynamical boundaries can be either a regular center, a singularity, or an infinity.

If this picture is correct, all singularities are connected: the kinematical singularity, plus possibly a dynamical singularity connected to it at either end. Similarly, all infinities are connected: the kinematical infinity, plus possibly a dynamical infinity connected to it at either end. This would rule out spacetimes with two or more disjoint singularities, for example cosmologies with a big bang and big crunch, or two or more infinities, for example Kruskal-like spacetimes [13].

We then focussed on SSHs: radial null geodesics that are invariant under the self-similarity. We investigated and classified a fairly generic family of SSHs by approximating them by a class of spacetimes we call the q-metric. We showed that these SSHs are not simply lines in the interior of the conformal diagram, but are T-shaped, with the crossbar of the T forming part of the either the kinematical infinity (fan) or kinematical singularity (splash). A neighbourhood of the SSH is therefore a triangle in the conformal diagram. The entire conformal diagram of a generic spherical DSS spacetime can be assembled from these extended SSHs. We have attempted a general classification, but our analysis is not rigorous, and we may have overlooked some kinematical possibilities. At least our framework fits all spherical DSS solutions known to us including some degenerate cases.

A problem arises in the q-metric with $q < -1$, where a part of what by dimensional analysis should be the kinematical infinity is in fact a non-scalar curvature singularity. However, these spacetimes have negative energy density, and imposing an energy condition saves the "one singularity, one infinity" result suggested by dimensional analysis.

Acknowledgments

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APPENDIX A: COORDINATE TRANSFORMATION NEAR THE SSH

Here we show how the standard form (19) of the metric near a SSH can be obtained. We assume that we have already used part of the gauge freedom (11) to set $x = 0$ along the SSH in order to obtain (18). We now carry out a further coordinate transformation of the form (11) which preserves this form of the metric, with new coefficients $a'(\tau)$, $b'(\tau')$, $c'(\tau')$ and $f'(\tau')$. We expand (11) and impose the condition that $O(x) = O(x')$. To leading order this gives

$$x' = \varphi_1(\tau)x + O(x^2), \quad \tau' = \tau + \psi_0(\tau) + \psi_1(\tau)x + O(x^2) \tag{A1}$$

Comparing coefficients, we find that

$$a = e^{-2\psi_0}\left[\varphi_1(1+\dot\psi_0)^2 a' + 2\dot\varphi_1(1+\dot\psi_0)b'\right], \tag{A2}$$
$$b = e^{-2\psi_0}\varphi_1(1+\dot\psi_0)b', \tag{A3}$$
$$c = e^{-2\psi_0}\left[\varphi_1^2 c' + 2\varphi_1\psi_1 b'\right], \tag{A4}$$
$$f = e^{-\psi_0}f' \tag{A5}$$

where $\dot\varphi_1 = d\varphi_1/d\tau$ and $\dot\psi_0 = d\psi_0/d\tau$. We now impose the gauge conditions

$$a' = 4q, \qquad b' = 1, \qquad c' = 0, \tag{A6}$$

where q is a constant to be determined. This gives the coupled ODEs

$$a = e^{-2\psi_0}\left[\varphi_1(1+\dot\psi_0)^2 4q + 2\dot\varphi_1(1+\dot\psi_0)\right], \tag{A7}$$
$$b = e^{-2\psi_0}\varphi_1(1+\dot\psi_0) \tag{A8}$$

for ψ_0 and φ_1. When these have been solved, there remains the explicit expression

$$c = e^{-2\psi_0}2\varphi_1\psi_1 \tag{A9}$$

for ψ_1 and the algebraic equation

$$f(\tau) = e^{-\psi_0}f'[\tau + \psi_0(\tau)] \tag{A10}$$

for $f'(\tau')$. After deleting the primes, we have (19).

Dividing (A7) by (A8), assuming that φ_1 and ψ_0 are periodic, and averaging, one obtains

$$4q = \overline{\left(\frac{a}{b}\right)}. \tag{A11}$$

This determines the constant q. In order to solve (A7) and (A8), note that they can be combined into a single first order linear ODE:

$$\frac{d}{d\tau}(\varphi_1 e^{-2\psi_0}) - 2\left(1 + \frac{a}{4b}\right)\varphi_1 e^{-2\psi_0} = -2(q+1)b. \tag{A12}$$

The equation $\dot y + fy = g$ with f and g periodic has a unique periodic solution y if and only if $\bar f \neq 0$ [18]. In our case this condition is $q \neq -1$. However, for $q = -1$ we have $g = 0$, and so there still is a solution that is unique up to an overall factor, $\ln y + \int f = \text{const}$. With $\varphi_1 e^{-2\psi_0}$ known, we can simply integrate

$$\dot\psi_0 = b\left(\varphi_1 e^{-2\psi_0}\right)^{-1} - 1 \tag{A13}$$

to obtain ψ_0 up to a constant. The condition for ψ_0 to be periodic is easily seen to be (A11) again. Because $\varphi_1 e^{-2\psi_0}$ is unique, the arbitrary additive constant in ψ_0 corresponds to a constant factor in φ_1. Together these changes leave (19) invariant up to an overall constant factor that can be thought of as a change of units (a change of l_0).

APPENDIX B: CURVATURE NEAR THE SSH

We have found that near the SSH the spacetime metric can be approximated by a metric of the form

$$ds^2 = -du\,dv + r(u,v)^2\,d\Omega^2. \tag{B1}$$

We define the null tetrad

$$E_1 = \frac{\partial}{\partial u}, \qquad E_2 = \frac{\partial}{\partial v},$$
$$E_3 = \frac{1}{r}\frac{\partial}{\partial\theta}, \qquad E_4 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{B2}$$

It is parallely transported along any radial geodesic. This is obvious because the spacetime is spherically symmetric and the reduced spacetime is flat. The only non-vanishing tetrad components of the Ricci tensor are

$$R_{11} = -\frac{2r_{,uu}}{r}, \tag{B3}$$

$$R_{12} = R_{21} = -\frac{2r_{,uv}}{r}, \tag{B4}$$

$$R_{22} = -\frac{2r_{,vv}}{r}, \tag{B5}$$

$$R_{33} = R_{44} = \frac{1+4r_{,u}r_{,v}+4rr_{,uv}}{r^2}. \tag{B6}$$

With

$$r(u,v) = f(puv^{q/p})^n v^{1/2p}, \tag{B7}$$

where f, q, $p \equiv q+1$ and n are constants, the non-vanishing frame coefficients are

$$R_{11} = -2n(n-1)u^{-2}, \tag{B8}$$

$$R_{12} = -2n\left(\frac{1}{2p}+n\frac{q}{p}\right)u^{-1}v^{-1}, \tag{B9}$$

$$R_{22} = -2\left(\frac{1}{2p}+n\frac{q}{p}\right)\left(\frac{1}{2p}+n\frac{q}{p}-1\right)v^{-2}, \tag{B10}$$

$$R_{33} = R_{44} = r^{-2} - 4R_{12}. \tag{B11}$$

The Ricci scalar is $R = -4R_{12} + 2R_{33}$, and the Kretschmann scalar is equal to R^2. We see that R_{33} and R always diverge at $r=0$. For $n \neq 0$, R_{11} or R_{12} also diverge at $u=0$. R_{22} diverges at $v=0$, except for two specific values of q. Note that one would miss the potential curvature singularities at $u=0$ and $v=0$ if one only looked for the blowup of curvature scalars.

APPENDIX C: BRADY CLASS I SPACETIMES

Starting from outgoing Bondi coordinates, Brady [11] defines the CSS coordinate $x = r/(-u)$. A natural choice of the log scale coordinate is $\tau = -\ln(-u)$, and this gives

$$ds^2 = e^{-2\tau}\left[-g(\bar{g}-2x)d\tau^2 - 2g\,dx\,d\tau + x^2\,d\Omega^2\right], \tag{C1}$$

where g and $\bar{g}$ are functions of x only. Brady's class I solutions have the range $0 \le x < \infty$. The dynamical boundary $x=0$ is a regular center, where $g, \bar{g} = 1 + O(x^2)$. As $x \to \infty$, $y \equiv \bar{g}/g \to 1/2$ and $z \equiv x/\bar{g} \to 1/(1+c)$ where $c \equiv \sqrt{4\pi\kappa} > 1$ is a parameter of the solution. The metric as $x \to \infty$ becomes

$$ds^2 \simeq e^{-2\tau}\left[-2(c^2-1)x^2 d\tau^2 - 4(1+c)x\,dx\,d\tau + x^2\,d\Omega^2\right]. \tag{C2}$$

This can be compactified by $x \equiv \tilde{x}^{-n}$ for any $n > 0$, which gives

$$ds^2 \simeq e^{-2\tau}\tilde{x}^{-2n-1}\Big[-2(c^2-1)\tilde{x}\,d\tau^2 + 4n(1+c)d\tilde{x}\,d\tau + \tilde{x}\,d\Omega^2\Big]. \tag{C3}$$

As $f_u \simeq (c-1)/(4n)\tilde{x}$, $\tilde{x}=0$ is a fan, but the compactification shows that it is also a dynamical null infinity.

[1] A. Ori and T. Piran, Phys. Rev. Lett. **59**, 2137 (1987).
[2] A. Ori and T. Piran, Phys. Rev. D. **42**, 1068 (1990).
[3] K. Lake, Phys. Rev. Lett. **68**, 3129 (1992).
[4] T. Harada and H. Maeda, Phys. Rev. D **63**, 084022 (2001).
[5] M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).
[6] A. M. Abrahams and C. R. Evans, Phys. Rev. Lett. **70**, 2980 (1993).
[7] C. R. Evans and J. S. Coleman, Phys. Rev. Lett. **72**, 1782 (1994).
[8] C. Gundlach, Living Reviews in Relativity **2**, 4 (1999); C. Gundlach, Phys. Rep. **376**, 339 (2003).
[9] D. Sornette, Phys. Rep. **297**, 239 (1998).
[10] M. D. Roberts, Gen. Rel. Grav. **21**, 907 (1989). See also Y. Oshiro, K. Nakamura, and A. Tomimatsu, Prog. Theor. Phys. **91**, 1265 (1994).
[11] P. R. Brady, Phys. Rev. D **51**, 4168 (1995). Our Fig. 4 is based on a personal communication from Brady.
[12] B. C. Nolan, Class. Quant. Grav. **18**, 1651 (2001).
[13] B. J. Carr and C. Gundlach, Phys. Rev. D **67**, 024035 (2003).
[14] J. M. Martín-García and C. Gundlach, Global structure of Choptuik's critical solution in scalar field collapse, gr-qc/0304070, to be published in Phys. Rev. D.
[15] B. J. Carr and A. A. Coley, Class. Quant. Grav. **16**, R31 (1999).
[16] K. Lake and T. Zannias, Phys. Rev. D **41**, 3866 (1990).
[17] S. A. Hayward, Class. Quant. Grav. **17**, 4021 (2000).
[18] C. Gundlach, Phys. Rev. D **55**, 695 (1997).